Renormalization of the driven sine-Gordon equation in 2+1 dimensions

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We renormalize the driven sine-Gordon model including the relevant $\lambda(\nabla h)^2/2$ nonlinearity of Kardar, Parisi, and Zhang [Phys. Rev. Lett. **56**, 889 (1986)]. We find that on a larger scale a nonzero λ is generated, even if initially $\lambda=0$, through an "interplay" of the pinning potential and the driving force.

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I. INTRODUCTION

As pointed out by Burton, Cabrera, and Frank [1] crystalline facets undergo a roughening transition. Below the roughening temperature T_R the facet has only rare thermal excitations. Height fluctuations are small and independent of the facet size. On the other hand, above T_R the height fluctuations increase logarithmically with the linear dimension. The nature of this phase transition was clarified in the 1970's through the mapping to the Coulomb gas [2] and the link to the exactly solved six-vertex model [3]. These investigations show that the roughening transition is in the Kosterlitz-Thouless universality class [4].

When using physically realistic parameters one notices that even in the rough phase the height fluctuations are small. Thus experimentally the transition can be observed only by indirect means. One beautiful set of such experiments has been carried out by Wolf *et al.* [5]; see also [6]. They consider solid ^{4}He in contact with the superfluid phase. The roughening temperature is $T_R=1.28$ K. In order to see the transition the superfluid pressure is increased beyond its equilibrium value. Thereby the superfluid freezes onto the facet and the helium crystal starts growing with some average velocity v. The growth mechanism depends sensitively on whether $T<T_R$ or $T>T_R$. For $T>T_R$ there are many sites at which the atoms in the fluid can stick to the crystal and the velocity depends approximately linearly on the driving force. On the other hand, for $T<T_R$ first a supercritical (two-dimensional) droplet has to be formed which then can grow to cover the whole facet. Thus growth is mostly layer by layer. For such an activated nucleation process the resulting growth velocity is exponentially small in the inverse driving force and hence greatly suppressed.

In order to analyze an experiment as the one of Wolf *et al.* one has to understand the kinetics of a *driven* interface close to the roughening transition. This is the subject of our paper. We will first explain a, surely oversimplified, theoretical model which serves as a convenient starting point for further investigations.

One standard approach to interface dynamics is in terms of an effective solid on solid (SOS) interface model. We ignore overhangs and assume that the instantaneous solid-superfluid interface is given through a height function $h(x)$, $x\in\mathbb{Z}^2$, taking integer values. In particular, this implies a simple cubic lattice structure of the crystal with lattice constant $a=1$, for simplicity. To each crystalline bond we associate a binding energy J. Then, up to a global constant, the interfacial energy is given by

$$H=J\sum_{\langle x,y\rangle}|h(x)-h(y)| , \tag{1}$$

where we sum only over pairs of nearest neighbors $\langle x,y\rangle$. In equilibrium this SOS model undergoes a roughening transition precisely in the form mentioned above.

We assume that atoms from the fluid phase may attach to the surface and that also atoms may desorb from the crystal. Thus we introduce rates c_x^+ (c_x^-) for the height variable $h(x)$ at x to increase (decrease) by one unit. In equilibrium the rates must satisfy detailed balance with respect to the Boltzmann weight with energy (1). In order to *drive* the interface the absorption processes are slightly favored as compared to desorption processes. Thus c_x^+ (c_x^-) are replaced by $c_x^{+\delta}=c_x^+(1+\delta)$ $[c_x^{-\delta}=c_x^-(1-\delta)]$. Thereby we lose detailed balance. The interface now has a nonzero upwards velocity v. We expect that in the moving frame of reference the interface acquires a statistically stationary state as $t\to\infty$. The theoretical problem is then to determine the growth velocity in dependence on the temperature and the bias (driving force).

In this form our problem looks hopelessly difficult. The best we can hope to do is to devise a renormalization-group (RG) scheme which should yield a decent description at least close to T_R and on scales large compared to the lattice constant. Lattice models such as (1) are not so well suited for renormalization. We therefore follow the standard route and average over small scales. Thus at time t the interface is given by a function $h(\mathbf{r},t)$ with $\mathbf{r}\in\mathbb{R}^2$, the two-dimensional plane. The lattice structure orthogonal to the facet is relevant and is kept in the form of an external cosine potential with period a. We then arrive at the driven sine-Gordon model

$$\eta\frac{\partial}{\partial t}h=\gamma\Delta h-\frac{2\pi}{a}V\sin\left[\frac{2\pi}{a}h\right]+F+R . \tag{2}$$

Let us explain the various terms. η is the inverse mobili-

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ty and fixes the time scale. γ is the surface stiffness, a is the lattice constant, and V is the strength of the pinning potential. F is the constant driving force. Finally, R is white noise with strength

$$\langle R(\mathbf{r},t)R(\mathbf{r}',t')\rangle = 2D\delta(\mathbf{r}-\mathbf{r}')\delta(t-t') . \tag{3}$$

If the driving force vanishes, then Eq. (2) has a unique stationary solution, namely, the equilibrium state

$$Z^{-1}\exp\left\{-\frac{1}{T}\int d^2r\left[\frac{\gamma}{2}|\nabla h(\mathbf{r})|^2 - V\cos\left[\frac{2\pi}{a}h(\mathbf{r})\right]\right]\right\} , \tag{4}$$

which is known as the static sine-Gordon model. By the fluctuation dissipation theorem the noise strength D is related to the temperature T by

$$D = \eta T . \tag{5}$$

The static sine-Gordon model also undergoes a roughening transition with the same critical behavior as the discrete SOS model. For $F > 0$ the solution to Eq. (2) will move with some average velocity v. Our goal is to determine v in dependence on the bare parameters of the model.

This problem was tackled by Nozières and Gallet [7], who devised a particular dynamic RG scheme. They make the obvious choice for the space of "coupling constants" by taking those appearing in (2) and (3). About the same time Kardar, Parisi, and Zhang (KPZ) [9] argued that for surface growth a term $(\lambda/2)(\nabla h)^2$ in Eq. (2) is *relevant* in the sense that it determines the large scale behavior of the moving interface. Their predictions have been confirmed for a large variety of growth models [10]. Thus from a systematic point of view we have to enlarge the space of coupling constants by including λ. Of course, the RG scheme will be only an approximation to the true RG flow in some infinite-dimensional space. However, this approximation should include at least all relevant parameters. In particular we expect that a term $(\lambda/2)(\nabla h)^2$ will be generated from (2) under renormalization.

In a recent Letter Hwa, Kardar, and Paczuski [11] in fact carried out the program just outlined. In their RG flow an initial $\lambda=0$ remains zero (at least to second order in V). This feature is rather surprising and it certainly contradicts the findings for the one-dimensional sine-Gordon chain [12], i.e., (2) with $r\in\mathbb{R}$. In the one-dimensional chain the strength of the effective KPZ nonlinearity can be computed in a good approximation and determines the large scale chain fluctuations. For example, starting with a flat profile $h(r,t=0)=0$, the width of the chain $[\langle h(r,t)^2\rangle - \langle h(r,t)\rangle^2]^{1/2}$ increases as $\lambda t^{1/3}$ for large t. Because of this contradiction we decided to reexamine the whole problem. Our RG flow is given in Eqs. (27) where we also discuss in more detail the relationship with Hwa, Kardar, and Paczuski.

This paper is outlined as follows. In Sec. II we apply the Nozières-Gallet RG scheme keeping systematically the term $(\lambda/2)(\nabla h)^2$. In Sec. III we discuss the extended RG flow. Close to T_R it is iterated numerically. We also discuss some predictions for the renormalized parameters and the importance of the $(\lambda/2)(\nabla h)^2$ term.

II. THE NOZIÈRES-GALLET RG SCHEME ENLARGED

Our starting point is the driven sine-Gordon equation

$$\eta\frac{\partial}{\partial t}h = \gamma\Delta h - \frac{2\pi}{a}V\sin\left[\frac{2\pi}{a}h\right] + \frac{\lambda}{2}(\nabla h)^2 + F + R \tag{6}$$

with the KPZ nonlinearity $(\lambda/2)(\nabla h)^2$ added on. The microscopic (bare) model is (6) with $\lambda=0$. Since we expect the $(\nabla h)^2$ term to be generated on a larger scale, we already included it in (6) on a provisional basis.

Physically fluctuations on a scale smaller than the lattice spacing a are suppressed because of the crystalline structure. This is implemented by cutting off the noise spectrum at $\Lambda=\pi/a$. Thus in momentum space the noise correlator is

$$\langle R(\mathbf{k},t)R(-\mathbf{k}',t')\rangle = 2Df\left[\frac{|\mathbf{k}|}{\Lambda}\right]\delta(\mathbf{k}-\mathbf{k}')\delta(t-t') . \tag{7}$$

As in [8] we choose a sharp cutoff function $f(x)=\theta(1-x)$.

We follow the scheme of [7] and [9] by first integrating over the momentum shell $\Lambda(1-dl) < |\mathbf{k}| \le \Lambda$. This can be done only perturbatively in λ and V and requires some work. In the second step we rescale back to the original Λ sphere in momentum space, i.e., $\mathbf{k}\to\mathbf{k}'=(1+dl)\mathbf{k}$, or equivalently $\mathbf{r}\to\mathbf{r}'=(1-dl)\mathbf{r}$. In order to keep the original lattice spacing we impose $h\to h'=h$ and thus $t\to t'=(1-2dl)t$. Therefore the various coefficients are rescaled as

$$\begin{aligned} &\eta\to\eta'=\eta,\quad \gamma\to\gamma'=\gamma,\quad \lambda\to\lambda'=\lambda,\\ &V\to V'=(1+2dl)V,\quad F\to F'=(1+2dl)F . \end{aligned} \tag{8}$$

In order to set up the perturbative scheme we rewrite Eq. (6) in a frame of reference moving with velocity F/η. Then

$$\eta\frac{\partial}{\partial t}h = \gamma\Delta h + \Phi(h) + R \tag{9}$$

with the perturbation

$$\Phi(h) = -\frac{2\pi}{a}V\sin\left[\frac{2\pi}{a}\left[h+\frac{F}{\eta}t\right]\right] + \frac{\lambda}{2}(\nabla h)^2 . \tag{10}$$

We now split R into $\bar{R}+\delta R$ where $\bar{R}$ contains only momenta $|\mathbf{k}|\le(1-dl)\Lambda$ and δR those in $(1-dl)\Lambda < |\mathbf{k}| \le \Lambda$. The height profile $h(\mathbf{r},t)$ depends on the complete history of the noise at earlier times $t'\le t$. We define $\bar{h}(\mathbf{r},t)$ to be the average of $h(\mathbf{r},t)$ with respect to the short wavelength components of the noise, i.e., with respect to δR,

$$\bar{h}(\mathbf{r},t) = \langle h(\mathbf{r},t)\rangle_{\delta R} ,$$

and denote the remainder by $\delta h = h - \bar{h}$. $\bar{h}$ satisfies some fairly complicated equation which is nonlocal in $(\mathbf{r},t)$. We will try to approximate it by an equation of the form (6) with suitably renormalized coefficients.

On both sides of (9) we average with respect to δR and obtain

$$\eta\frac{\partial}{\partial t}\bar{h}=\gamma\Delta\bar{h}+\bar{\Phi}(\bar{h},\delta h)+\bar{R}\ ,$$
$$\eta\frac{\partial}{\partial t}\delta h=\gamma\Delta\delta h+\delta\Phi(\bar{h},\delta h)+\delta R\ . \quad (11)$$

Here $\bar{\Phi}=\langle\Phi\rangle_{\delta R}$ and the second equation is simply the difference between the first one and (9). Since δh is infinitesimally small, we have

$$\bar{\Phi}=-\frac{2\pi}{a}V\sin\left[\frac{2\pi}{a}\left[\bar{h}+\frac{F}{\eta}t\right]\right]\left[1-\frac{2\pi^2}{a^2}\langle(\delta h)^2\rangle_{\delta R}\right]+\frac{\lambda}{2}(\nabla\bar{h})^2+\frac{\lambda}{2}\langle(\nabla\delta h)^2\rangle_{\delta R}\ ,$$
$$\delta\Phi=-\frac{4\pi^2}{a^2}V\cos\left[\frac{2\pi}{a}\left[\bar{h}+\frac{F}{\eta}t\right]\right]\delta h+\lambda\nabla\bar{h}\cdot\nabla\delta h\ . \quad (12)$$

To this order $\bar{\Phi}$ depends on $\delta h(\mathbf{r},t)$ only through the averages $\langle(\delta h)^2\rangle$ and $\langle(\nabla\delta h)^2\rangle$. We must approximate them by expressions local in $(\mathbf{r},t)$. In (11) we regard $\delta\Phi$ as a perturbation and expand δh to first order as $\delta h=\delta h^{(0)}+\delta h^{(1)}+\cdots$. Then

$$\delta h^{(0)}(\mathbf{r},t)=\int_{-\infty}^{t}dt'\int d^2r'\chi_0(\mathbf{r}-\mathbf{r}',t-t')\delta R(\mathbf{r}',t')\ ,$$
$$\delta h^{(1)}(\mathbf{r},t)=\int_{-\infty}^{t}dt'\int d^2r'\chi_0(\mathbf{r}-\mathbf{r}',t-t')\times\left[-\frac{4\pi^2}{a^2}V\cos\left[\frac{2\pi}{a}\left[\bar{h}(\mathbf{r}',t')+\frac{F}{\eta}t'\right]\right]\delta h^{(0)}(\mathbf{r}',t')+\lambda\nabla\bar{h}(\mathbf{r}',t')\cdot\nabla\delta h^{(0)}(\mathbf{r}',t')\right]\ . \quad (13)$$

$\chi_0(\rho,\tau)=1/(4\pi\gamma\tau)\exp[-\eta\rho^2/(4\gamma\tau)]$ is the propagator of the dynamics of the unperturbed Gaussian surface in two dimensions. Note that we choose as initial condition a flat surface at $t=-\infty$, i.e., $h(\mathbf{r},-\infty)\equiv 0$. The averages in the definition of $\bar{\Phi}$ are given approximately by

$$\langle[\delta h(\mathbf{r},t)]^2\rangle=\langle[\delta h^{(0)}(\mathbf{r},t)]^2\rangle+2\langle\delta h^{(0)}(\mathbf{r},t)\delta h^{(1)}(\mathbf{r},t)\rangle\ ,$$
$$\langle[\nabla\delta h(\mathbf{r},t)^2\rangle=\langle[\nabla\delta h^{(0)}(\mathbf{r},t)]^2\rangle+2\langle\nabla\delta h^{(0)}(\mathbf{r},t)\cdot\nabla\delta h^{(1)}(\mathbf{r},t)\rangle\ . \quad (14)$$

To proceed further we need the following correlation:

$$\langle\delta h^{(0)}(\mathbf{r},t)\delta h^{(0)}(\mathbf{r}',t')\rangle=\frac{T}{2\pi\gamma}J_0(\Lambda|\mathbf{r}-\mathbf{r}'|)\exp\left[-\frac{\gamma\Lambda^2(t-t')}{\eta}\right]dl\ , \quad (15)$$

where J_0 is the zeroth order Bessel function [8]. By differentiating with respect to $\mathbf{r}$ and $\mathbf{r}'$ we obtain the correlations containing $\nabla\delta h$,

$$\langle\partial_i\delta h^{(0)}(\mathbf{r},t)\partial_j\delta h^{(0)}(\mathbf{r}',t')\rangle=dl\frac{T}{2\pi\gamma}\Lambda^2\exp\left[-\frac{\gamma\Lambda^2(t-t')}{\eta}\right]\left[\tfrac{1}{2}[J_0(\Lambda|\mathbf{r}-\mathbf{r}'|)+J_2(\Lambda|\mathbf{r}-\mathbf{r}'|)]\delta_{ij}-J_2(\Lambda|\mathbf{r}-\mathbf{r}'|)\frac{(r_i-r_i')(r_j-r_j')}{(\mathbf{r}-\mathbf{r}')^2}\right]\ , \quad (16)$$

where we use $\partial_i h$ as convenient shorthand for $\partial h/\partial r_i$. At this stage we can already calculate the first order corrections to $\bar{\Phi}$. The average $\langle[\delta h^{(0)}(\mathbf{r},t)]^2\rangle$ changes the prefactor of the pinning force and therefore renormalizes V. $\langle[\nabla\delta h^{(0)}(\mathbf{r},t)]^2\rangle$ is a constant term which therefore renormalizes the driving force F. We have then

$$dV^{(1)}=-\frac{2\pi^2}{a^2}\langle[\delta h^{(0)}(\mathbf{r},t)]^2\rangle=-\frac{\pi T}{a^2\gamma}V\,dl\ ,$$
$$dF^{(1)}=\frac{\lambda}{2}\langle[\nabla\delta h^{(0)}(\mathbf{r},t)]^2\rangle=\frac{\lambda T}{4\pi\gamma}\Lambda^2 dl\ . \quad (17)$$

Adjusting our notation to that of Nozières and Gallet we redefine $U=V/\Lambda^2$ and $K=F/\Lambda^2$. To first order they satisfy the flow equations

$$\frac{dU}{dl}=\left[2-\frac{\pi T}{\alpha^2\gamma}\right]U,\quad \frac{dK}{dl}=2K+\frac{T}{4\pi\gamma}\lambda \tag{18}$$

with the rescaling (8) already included.

Let us now take into account the corrections of second order by evaluating $\langle \delta h^{(0)}\delta h^{(1)}\rangle$ and $\langle \nabla\delta h^{(0)}\cdot\nabla\delta h^{(1)}\rangle$. Using (13) we obtain

$$\begin{aligned}\langle \delta h^{(0)}(\mathbf{r},t)\delta h^{(1)}(\mathbf{r},t)\rangle &= -\frac{4\pi^2}{a^2}V\int_{-\infty}^{t}dt'\int d^2r'\cos\left[\frac{2\pi}{a}\left[\bar{h}(\mathbf{r}',t')+\frac{F}{\eta}t'\right]\right]\chi_0(\mathbf{r}-\mathbf{r}',t-t')\langle \delta h^{(0)}(\mathbf{r},t)\delta h^{(0)}(\mathbf{r}',t')\rangle \\ &\quad +\lambda\sum_{i=1}^{2}\int_{-\infty}^{t}dt'\int d^2r'\partial_i\bar{h}(\mathbf{r}',t')\chi_0(\mathbf{r}-\mathbf{r}',t-t')\langle \delta h^{(0)}(\mathbf{r},t)\partial_i\delta h^{(0)}(\mathbf{r}',t')\rangle = A_1+A_2\end{aligned} \tag{19}$$

and

$$\begin{aligned}&\langle \nabla\delta h^{(0)}(\mathbf{r},t)\cdot\nabla\delta h^{(1)}(\mathbf{r},t)\rangle \\ &= -\frac{4\pi^2}{a^2}V\frac{\eta}{2\gamma}\sum_{i=1}^{2}\int_{-\infty}^{t}dt'\int d^2r'\cos\left[\frac{2\pi}{a}\left[\bar{h}(\mathbf{r}',t')+\frac{F}{\eta}t'\right]\right]\left[-\frac{r_i'-r_i}{t-t'}\right]\chi_0(\mathbf{r}-\mathbf{r}',t-t')\langle \partial_i\delta h^{(0)}(\mathbf{r},t)\delta h^{(0)}(\mathbf{r}',t')\rangle \\ &\quad +\frac{\eta\lambda}{2\gamma}\sum_{i,j=1}^{2}\int_{-\infty}^{t}dt'\int d^2r'\left[-\frac{r_i'-r_i}{t-t'}\right]\partial_j\bar{h}(\mathbf{r}',t')\chi_0(\mathbf{r}-\mathbf{r}',t-t')\langle \partial_i\delta h^{(0)}(\mathbf{r},t)\partial_j\delta h^{(0)}(\mathbf{r}',t')\rangle \\ &= A_3+A_4\,.\end{aligned} \tag{20}$$

The terms denoted by A_2 and A_3 contribute to the renormalization by a term of the form const$\times\lambda V$. If we consider a translation of the system by half a lattice spacing $h\rightarrow h'=h+a/2$, then in the sine-Gordon equation merely the sign of V changes. Clearly, the RG flow has to respect this invariance under the transformation $V\rightarrow V'=-V$. In particular, terms of the form const$\times\lambda V$ are ruled out. This argument does not apply to the equation for V itself. But in comparison to the first order term we may neglect the second order term anyhow. Thus altogether A_2 and A_3 are dropped.

In order to understand the term A_4 we set the periodic pinning force in Eq. (6) equal to zero, i.e., $V=0$. We then obtain the KPZ equation whose RG flow is well studied [9,13]. For our rescaling scheme, due to the KPZ nonlinearity we find the following correction:

$$d\gamma^{\mathrm{KPZ}}=0,\quad d\lambda^{\mathrm{KPZ}}=0,\quad dD^{\mathrm{KPZ}}=dl\frac{1}{8\pi}\frac{\lambda^2 D}{\gamma^3}D\,. \tag{21}$$

In addition we have $d\eta^{\mathrm{KPZ}}=0$ and $dF^{\mathrm{KPZ}}=0$, since these coefficients can be absorbed through the transformations $t\rightarrow t'=t/\eta$ and $h\rightarrow h'=h-Ft/\eta$, and $dV^{\mathrm{KPZ}}=0$ of course.

Finally we consider the term A_1 which contributes to the renormalization through a mode coupling caused by the pinning force. The contribution of A_1 to $\bar{\Phi}$ is

$$\begin{aligned}\Phi^{\mathrm{SG}} &= -\frac{32\pi^5 V}{a^5}\int_{-\infty}^{t}dt'\int d^2r'\chi_0(\mathbf{r}-\mathbf{r}',t-t') \\ &\quad\times\sin\left[\frac{2\pi}{a}\left[\bar{h}(\mathbf{r},t)+\frac{F}{\eta}t\right]\right]\cos\left[\frac{2\pi}{a}\left[\bar{h}(\mathbf{r}',t')+\frac{F}{\eta}t'\right]\right]\langle \delta h^{(0)}(\mathbf{r},t)\delta h^{(0)}(\mathbf{r}',t')\rangle\,.\end{aligned} \tag{22}$$

The product of sine and cosine is split as $\sin x\cos y=[\sin(x+y)+\sin(x-y)]/2$. The term $\sin(x+y)$ is neglected since it contributes only to higher harmonics in the pinning potential which become irrelevant under renormalization. This can be seen from Eq. (18). Inserting the explicit expressions for $\langle \delta h^{(0)}(\mathbf{r},t)\delta h^{(0)}(\mathbf{r}',t')\rangle$ and $\chi_0(\mathbf{r}-\mathbf{r}',t-t')$ we approximate Eq. (22) by

$$\begin{aligned}&-\frac{2\pi^3V^2T}{\gamma^2a^5}dl\int_{-\infty}^{t}dt'\int d^2r'\frac{1}{t-t'}\exp\left[-\frac{\eta(\mathbf{r}-\mathbf{r}')^2}{4\gamma(t-t')}-\frac{\gamma}{\eta}\Lambda^2(t-t')\right] \\ &\quad\times J_0(|\mathbf{r}-\mathbf{r}'|\Lambda)\sin\left[\frac{2\pi}{a}\left[\bar{h}(\mathbf{r},t)-\bar{h}(\mathbf{r}',t')+\frac{F}{\eta}(t-t')\right]\right]\,.\end{aligned}$$

In the next step we split the sine as

$$\sin\left[\frac{2\pi}{a}\left[\bar h(\mathbf r,t)-\bar h(\mathbf r',t')+\frac{F}{\eta}(t-t')\right]\right]$$
$$=\sin\left[\frac{2\pi}{a}[\bar h(\mathbf r,t)-\bar h(\mathbf r',t')]\right]\cos\left[\frac{2\pi}{a}\frac{F}{\eta}(t-t')\right]+\cos\left[\frac{2\pi}{a}[\bar h(\mathbf r,t)-\bar h(\mathbf r',t')]\right]\sin\left[\frac{2\pi}{a}\frac{F}{\eta}(t-t')\right]. \quad (23)$$

Nozières and Gallet provide us with the correct approximations to (23) by terms local in $(\mathbf r,t)$, namely,

$$\sin\left[\frac{2\pi}{a}[\bar h(\mathbf r,t)-\bar h(\mathbf r',t')]\right]=\left\langle\cos\left[\frac{2\pi}{a}[\bar h(\mathbf r,t)-\bar h(\mathbf r',t')]\right]\right\rangle_{\bar R}\times\frac{2\pi}{a}\left[\frac{\partial}{\partial t}\bar h(\mathbf r,t)(t-t')-\frac{1}{2}\partial_i\partial_j\bar h(\mathbf r,t)(r_i-r_i')(r_j-r_j')+\cdots\right],$$

$$\cos\left[\frac{2\pi}{a}[\bar h(\mathbf r,t)-\bar h(\mathbf r',t')]\right]=\left\langle\cos\left[\frac{2\pi}{a}[\bar h(\mathbf r,t)-\bar h(\mathbf r',t')]\right]\right\rangle_{\bar R}\left[1-\frac{2\pi^2}{a^2}[\partial_i\bar h(\mathbf r,t)]^2(r_i-r_i')^2+\cdots\right].$$

Note that terms like $\nabla\bar h\cdot(\mathbf r-\mathbf r')$ vanish as we integrate them over d^2r'. The average $\langle\cos\{(2\pi/a)[\bar h(\mathbf r,t)-\bar h(\mathbf r',t')]\}\rangle_{\bar R}$ is easily evaluated, since we only need it to zeroth order in $\bar\Phi$. The distribution of $\bar h(\mathbf r,t)-\bar h(\mathbf r',t')$ is then Gaussian and we find

$$\left\langle\cos\left[\frac{2\pi}{a}[\bar h(\mathbf r,t)-\bar h(\mathbf r',t')]\right]\right\rangle_{\bar R}=\exp\left[-\frac{2\pi^2}{a^2}\langle[\bar h(\mathbf r,t)-\bar h(\mathbf r',t')]^2\rangle_{\bar R}\right]$$

and

$$\langle[\bar h(\mathbf r,t)-\bar h(\mathbf r',t')]^2\rangle_{\bar R}=\frac{T}{\pi\gamma}\int_0^\Lambda\frac{dk}{k}\{1-J_0[k|\mathbf r-\mathbf r'|)]\}\exp\left[-\frac{\gamma}{\eta}k^2(t-t')\right]=\frac{T}{\pi\gamma}\varphi(|\mathbf r-\mathbf r'|,t-t'). \quad (24)$$

Thus (22) is approximated by

$$-\frac{2\pi^3V^2T}{\gamma^2a^5}dl\int_{-\infty}^{t}dt'\int d^2r'\frac{1}{t-t'}J_0(|\mathbf r-\mathbf r'|\Lambda)\exp\left[-\frac{\eta(\mathbf r-\mathbf r')^2}{4\gamma(t-t')}-\frac{\gamma}{\eta}\Lambda^2(t-t')-2\frac{\pi T}{a^2\gamma}\varphi(|\mathbf r-\mathbf r'|,t-t')\right]$$
$$\times\left[\frac{2\pi}{a}\left[\frac{\partial}{\partial t}\bar h(\mathbf r,t)(t-t')-\frac{1}{2}\partial_i\partial_j\bar h(\mathbf r,t)(r_i-r_i'(r_j-r_j')\right]\cos\left[\frac{2\pi}{a}\frac{F}{\eta}(t-t')\right]\right.$$
$$\left.+\left[1-\frac{2\pi^2}{a^2}[\partial_i\bar h(\mathbf r,t)]^2(r_i-r_i')^2\right]\sin\left[\frac{2\pi}{a}\frac{F}{\eta}(t-t')\right]\right]. \quad (25)$$

We recognize terms proportional to $\partial\bar h/\partial t,\Delta\bar h$, and $(\nabla\bar h)^2$, which renormalize η, γ, and λ respectively. Furthermore there is a constant term which renormalizes F. The latter two terms do not appear in the RG scheme of Nozières and Gallet since they restrict their renormalization to η, γ, and V. Note that these terms vanish for zero driving force because of the prefactor $\sin[2\pi F(t-t')/(a\eta)]$. The physical meaning of this property will be discussed in Sec. III.

In order to evaluate (25) we transform $\mathbf r-\mathbf r'$ to polar coordinates (ρ,ϑ). We integrate over $d\vartheta$ and write the integral in terms of the dimensionless quantities $\tilde\rho\equiv\Lambda\rho$ and $x\equiv\gamma(t-t')/(\eta\rho^2)$. The renormalization due to the sine-Gordon mode coupling is then

$$\begin{aligned}
d\eta^{\mathrm{SG}}&=\frac{8\pi^4\eta}{\gamma^2a^4}U^2dl\frac{\pi T}{\gamma a^2}\int_0^\infty dx\int_0^\infty d\tilde\rho\,\tilde\rho J_0(\tilde\rho)\exp\left[-\frac{1}{4x}-x\tilde\rho^2-2\frac{\pi T}{\gamma a^2}\varphi(\tilde\rho,x)\right]\cos\left[\frac{2\pi}{a}\frac{Kx\tilde\rho^2}{\gamma}\right],\\
d\gamma^{\mathrm{SG}}&=\frac{2\pi^4}{\gamma a^4}U^2dl\frac{\pi T}{\gamma a^2}\int_0^\infty\frac{dx}{x}\int_0^\infty d\tilde\rho\,\tilde\rho^3J_0(\tilde\rho)\exp[\dots]\cos\left[\frac{2\pi}{a}\frac{Kx\tilde\rho^2}{\gamma}\right],\\
d\lambda^{\mathrm{SG}}&=\frac{8\pi^5}{\gamma a^5}U^2dl\frac{\pi T}{\gamma a^2}\int_0^\infty\frac{dx}{x}\int_0^\infty d\tilde\rho\,\tilde\rho^3J_0(\tilde\rho)\exp[\cdots]\sin\left[\frac{2\pi}{a}\frac{Kx\tilde\rho^2}{\gamma}\right],\\
dK^{\mathrm{SG}}&=\frac{4\pi3}{\gamma a^3}U^2dl\frac{\pi T}{\gamma a^2}\int_0^\infty\frac{dx}{x}\int_0^\infty d\tilde\rho\,\tilde\rho J_0(\tilde\rho)\exp[\cdots]\sin\left[\frac{2\pi}{a}\frac{Kx\tilde\rho^2}{\gamma}\right].
\end{aligned}\quad (26)$$

The integrals in (26) depend on T, K, and γ, more precisely on the parameters $n=\pi T/(\gamma a^2)$ and $\kappa=2\pi K/(a\gamma)$ in the arguments of the exponential and the sine or cosine, respectively. We write $A^{(\eta)}(n;\kappa)$, $A^{(\gamma)}(n;\kappa)$, $A^{(\lambda)}(n;\kappa)$, and $A^{(K)}(n;\kappa)$ as shorthand for the four integrals in (26). The complete renormalization-group flow equations to second order are then

$$\begin{aligned}
\frac{dU}{dl}&=(2-n)U\ ,\\
\frac{d\gamma}{dl}&=\frac{2\pi^4}{\gamma a^4}nA^{(\gamma)}(n;\kappa)U^2\ ,\\
\frac{d\eta}{dl}&=\frac{8\pi^4}{\gamma a^4}\frac{\eta}{\gamma}nA^{(\eta)}(n;\kappa)U^2\ ,\\
\frac{d\lambda}{dl}&=\frac{8\pi^5}{\gamma a^5}nA^{(\lambda)}(n;\kappa)U^2\ ,\\
\frac{dD}{dl}&=\frac{1}{8\pi}D\frac{D\lambda^2}{\gamma^3}+\frac{8\pi^4}{\gamma a^4}\frac{D}{\gamma}nA^{(\eta)}(n;\kappa)U^2\ ,\\
\frac{dK}{dl}&=2K+\frac{D}{4\eta\pi\gamma}\lambda-\frac{4\pi^3}{\gamma a^3}nA^{(K)}(n;\kappa)U^2\ .
\end{aligned}\tag{27}$$

III. PROPERTIES OF THE RG FLOW: PHYSICAL RESULTS

Our main result can be deduced from the equation for $d\lambda/dl$ (27). Even if initially $\lambda=0$, a nonzero λ is generated through the lattice potential for the case of a driven interface. Once $\lambda\neq 0$, the effective temperature will increase through further renormalization. At some point $\pi T/(\gamma a^2)=n>2$, which by (27) implies that U decreases to zero. Thus on a large scale the effect of the lattice potential vanishes and the moving interface is kinetically rough with a scaling behavior governed by the KPZ equation.

We note a few consistency checks. If $V=0$, then Eqs. (27) reduce to the RG flow of the KPZ equation, essentially by construction. If we fix $\lambda=0$ then Eqs. (27) become the RG flow of Nozières and Gallet. Furthermore, if initially $\lambda=0$ and $K=0$, i.e., zero driving force, these parameters remain zero under the RG flow because $A^{(\lambda)}(2;0)=A^{(K)}(2;0)=0$. By (27) also $T=D/\eta=$const and in the (U,γ,η) subspace there is the line of fixed points $U=0, \pi T/(\gamma a^2)=2$, physically corresponding to the roughening transition. The RG flow for this equilibrium transition is discussed in [7].

It is not very profitable to discuss the RG flow (27) in full generality. Also (27) stops to be valid at very low temperatures. A natural approach is to study the flow close to the line of fixed points $U=0$, $\lambda=0$, and $\pi T/(\gamma a^2)=2$ which corresponds to setting $n=2$ in Eqs. (27). Then it is convenient to define the reduced variables

$$t=\frac{2\pi D}{\eta a^2\gamma}-4,\quad \bar U=\frac{4\pi^2}{a^2}\sqrt{A}\frac{U}{\gamma},\quad \bar\lambda=\frac{a}{\pi}\sqrt{\eta}\frac{\lambda}{\gamma}\ ,\tag{28}$$

where we have set $A=A^{(\gamma)}(2;0)$. Note that up to a constant prefactor $\bar\lambda$ is the effective coupling constant in [9] within our fixed point approximation. In these new variables the RG flow equations become

$$\begin{aligned}
\frac{d\gamma}{dl}&=\frac{1}{4}\frac{A^{(\gamma)}(2;\kappa)}{A}\bar U^2\gamma\ ,\\
\frac{d\eta}{dl}&=\frac{A^{(\eta)}(2;\kappa)}{A}\bar U^2\eta\ ,\\
\frac{d\bar U^2}{dl}&=-\bar U^2\left[t+\frac{A^{(\gamma)}(2;\kappa)}{2A}\bar U^2\right]\ ,\\
\frac{d\lambda}{dl}&=\frac{A^{(\eta)}(2;\kappa)-\frac{1}{2}A^{(\gamma)}(2;\kappa)}{2A}\bar U^2\lambda+\frac{A^{(\lambda)}(2;\kappa)}{A}\sqrt{\eta}\bar U^2\ ,\\
\frac{dt}{dl}&=\bar\lambda^2-\frac{A^{(\gamma)}(2;\kappa)}{A}\bar U^2\ .
\end{aligned}\tag{29}$$

Here we approximate the equation for dK/dl by setting $dK/dl=2K$, i.e., $K=K_0e^{2l}$, and therefore K no longer appears explicitly in Eqs. (29).

In order to iterate (29), which will be done numerically, we first evaluate the integrals $A(2;\kappa)$. These are two-dimensional integrals. The integrands consist of a product of an oscillatory function, $\cos(\kappa x\tilde\rho^2)$ or $\sin(\kappa x\tilde\rho^2)$, and a smooth function which is concentrated around one peak approximately at $\tilde\rho=x=1$ and decays exponentially fast outside. $A^{(\gamma)}$ and $A^{(\eta)}$ are maximal for $\kappa=0$ and decay slowly as κ increases. When $\kappa>2$ the fast oscillations of $\cos(\kappa x\tilde\rho^2)$ already lie inside the central peak. The integrals almost vanish. The properties of $A^{(\lambda)}$ are similar, but $A^{(\lambda)}(2;0)=0$.

In Fig. 1 we sketch a numerical evaluation of the A coefficients. If the interface is driven, κ increases approximately as e^{2l} and the A coefficients vanish on large scales. Thus the sine-Gordon coupling loses its influence and the further rescaling is governed by the RG flow of the KPZ equation. This behavior has a simple physical interpretation: The moving surface crosses maxima and minima of the pinning potential. On a time scale sufficiently coarse such that the crossing of one lattice spacing cannot be resolved, one will only see the average pinning force, which is zero.

We have iterated (29) numerically we infer typical values for the parameters in Eqs. (29) as they appear in the ^{4}He experiments of [6] and [14]: The roughening temperature is $T_R=1.28$ K $=k_B^{-1}2\gamma a^2/\pi$, where γ

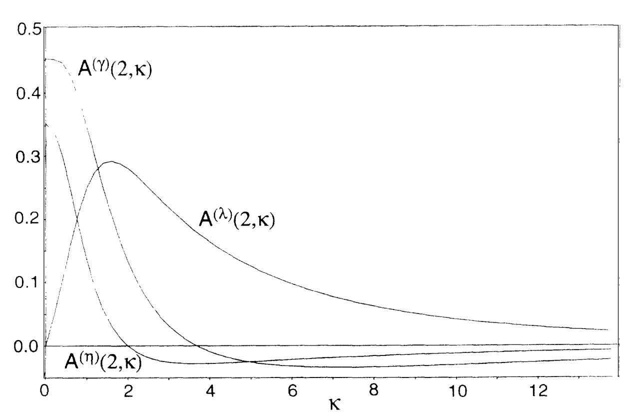

FIG. 1. The integrals in the RG flow equations versus κ.

is the macroscopic surface tension. Tracing back the critical trajectory we obtain for the bare parameters $\bar{U}_0^2=\exp[\pi k_B T_R/(\gamma_0 a^2)-2]-4\pi k_B T_R/(a^2\gamma_0)+4$. Thus the initial value of the temperaturelike parameter t can be fitted to the real temperature by $T=T_R(4+t_0)/(4+\sqrt{2}U_0)$. Typical initial values for the dimensionless parameter κ, corresponding to the strength of the driving force, are $\kappa<10^{-6}$.

In Fig. 2 we show the mobility of the interface for different values of the driving force. For $\kappa_0=0$, μ denotes the coefficient of the linear response of the growth velocity v to an external driving force F, i.e., $\mu=\lim_{F\to 0}(v/F)$. We recover the roughening transition. The mobility vanishes discontinuously at $T=T_R$. For $\kappa>0$, μ denotes a somewhat different quantity which for simplicity we call mobility as well: the change of the growth velocity under variation of the driving force, dv/dF at $F>0$. It decays smoothly as T decreases, in a driven system the roughening transition is blurred. Note also that for $T\to\infty$ the mobility approaches the value of the free Gaussian surface.

To understand the importance of the KPZ nonlinearity we iterated the flow (27) with $\bar{\lambda}=0$ held fixed. It turns out that the relative error is of the order 10^{-3} and even smaller. The point is that on the scale where the sine-Gordon coupling is wiped out by the moving interface the effective KPZ nonlinearity is still very small.

To see an effect we would have to go to much larger scales. Nattermann and Tang [15] give an estimate of the crossover length ξ_c. If ξ_0 is the length scale belonging to the values of $\bar{\lambda}$ in Fig. 3, then $\xi_c=\xi_0\exp(8\pi/\bar{\lambda}^2)$. For distances less than ξ_c the interface is logarithmically rough as an equilibrium interface. Only beyond ξ_c the fluctuations are stronger and are governed by the strong coupling fixed point of the KPZ equation which predicts a roughness exponent $\zeta\approx 0.39$. These findings agree with the argument of Balibar and Bouchaud [14], who using rough estimates obtained from the observed surface structure claim the minor importance of a KPZ term for the description of the ^{4}He experiments.

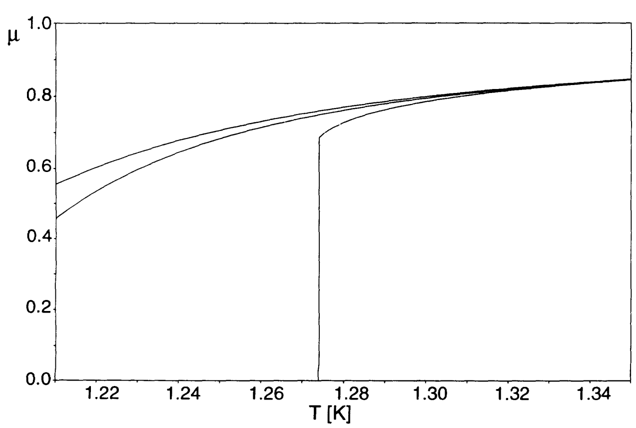

FIG. 2. The mobility of the interface for different values of the driving force. The lowest curve belongs to the undriven surface $\kappa_0=0$. The jump in μ corresponds to the roughening transition. The other two curves show μ for the driven SG model: the lower one for $\kappa_0=10^{-7}$ and the upper one for $\kappa_0=10^{-6}$.

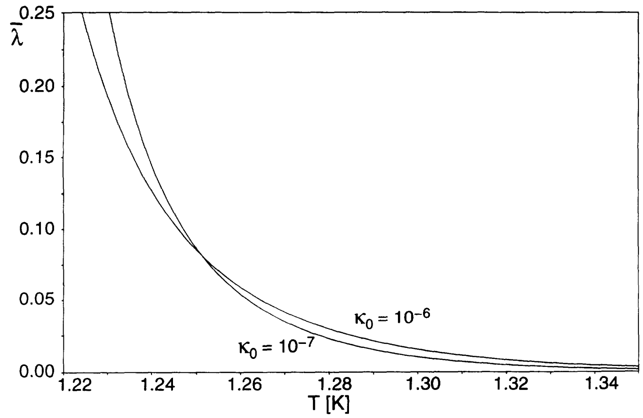

FIG. 3. The effective KPZ nonlinearity $\bar{\lambda}$ versus temperature for $\kappa_0=10^{-7}$ and 10^{-6}.

We remind the reader that the RG flow (27) is valid only for small pinning potential (since we expanded in V). Thus our prediction, namely, a small effective λ, can be valid only in the same range. For a strong pinning potential there is no systematic theory. In approximation we expect the following growth mechanism. If $h=0$ initially, then there is a certain rate to form a droplet in the first layer $\{h=a\}$. Once formed the droplet will expand with uniform speed (depending on F). Since droplets are formed independently, this noise causes second layer droplets to be already present before the first layer is completed. After several layers a rough surface morphology develops. The nucleation rate and lateral growth speed can be estimated [16]. An understanding of the statistical aspects is missing. But it is not unlikely that the KPZ nonlinear effects are much more pronounced than those found here. In the same spirit the lattice model explained in the Introduction could have a large effective coupling. We are not aware of a Monte Carlo study of this point, although it is certainly in reach with current machines.

IV. CONCLUSION

In this paper we treated the dynamic behavior of the driven sine-Gordon model in 2+1 dimensions. We enlarged the RG analysis of Nozières and Gallet to include terms proportional to $(\nabla h)^2$ and thus obtained RG flow equations to the second order in the strength of the pinning potential and the KPZ nonlinearity. As in Hwa, Kardar, and Paczuski [11] the presence of a KPZ nonlinearity causes the pinning potential to vanish on large scales. However, we propose a distinct mechanism: At microscopic scale the model consists of a Gaussian surface pinned by a cosine potential. Therefore a natural choice is to take $\lambda=0$ for the bare model. The pinning potential and the driving force together generate an effective $\lambda>0$ on larger scales. In our interpretation kinetic roughening is caused by the fact that the surface is driven, in contrast to [11] where a $\lambda > 0$ is put in by hand.

The presence of a KPZ nonlinearity turns out to be negligible on scales typical for experiments, if we take systems with weak crystal pinning potential as the interfaces between solid ^{4}He and its superfluid melt. In these cases no KPZ-like correction to the theory of Nozières and Gallet is required.

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